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THIN VISCOUS ELLIPTICAL ACCRETION DISCS WITH ORBITS SHARING A COMMON LONGITUDE OF PERIASTRON II. POLYNOMIAL SOLUTIONS TO THE DYNAMICAL EQUATION FOR INTEGER VALUES OF THE POWERS IN THE VISCOSITY LAW

Dimitar Dimitrov

Space Research Institute - Bulgarian Academy of Sciences

Abstract

For integer exponents n (n = -1, 0, 1, 2 and 3) in the viscosity law $\eta = \beta\,\Sigma^{n}$ with Σ – surface density of the accretion disc, we have investigated the polynomial approach to the solutions of the dynamical equation in the accretion disc model of Lyubarskij et al. [12]. Power series expansions of the eccentricity $e(u)$, its derivatives $\dot{e}(u)$ and $\ddot{e}(u)$, their powers $e^2(u)$, $e^3(u)$,...; $\dot{e}^2(u)$, $\dot{e}^3$, etc. are truncated at appropriate values of the exponents and then substituted in the dynamical equation. Making additional truncations in the intermediate products and results, we have nevertheless achieved accuracy of the solution better than ~ 10% for large enough domains in the plane (e, $\dot{e}$). These results are established graphically by comparing the polynomial approximation of the eccentricity $e_{polynomial}$ (where $u \equiv \ln p$; p is the focal parameter of the ellipse) and the exact values $e_{exact}(u)$ derived by means of numerical solving of the equation. The solutions of the second order ordinary differential equation of motion are parameterized by means of the boundary conditions $e_0 \equiv e(u = 0)$ and $\dot{e}_0 \equiv \dot{e}(u = 0)$. The coefficients in the power series expansions are evaluated in explicit form, but because of their complexity we give only their "sizes", considered as lengths of the files which represent them. Problems referred to the possible singularities of the results are also discussed.

Introduction

Both analytical and numerical studies during the last two decades reveal many of the observational characteristics of the accretion discs around compact objects with high masses (black holes in the active galactic nuclei) and stellar masses (white dwarfs, neutron stars and black holes in binary stellar systems).The comparison between the theoretical models and the observations provides to extract information not only about the physical properties of these objects but also about the structure and physical conditions in the accretion discs themselves. In particular, it is evident that approximately half of the young stellar objects, recognized as binaries, are associated with geometrically thin and optically thick circular accretion discs. For such cases the geometry and physics of the accretion flows are considerably complicated in comparison with the identical phenomena around single compact bodies. Similar situations may arise when a planet system forms inside the disc and the protoplanets locally deplete the material along their orbits. The recent observational evidence about the existing of extrasolar planet systems have given rise to another aspect concerning the radial structure of the accretion discs. As a rule, the suspected extrasolar planets have orbits with high eccentricities. Although the eccentricity values and also the radii of the protoplanet orbits may change for different stages of the planet-forming evolution process [1], it is suggested that the elongated eccentric orbits arise because of the eccentric shapes of the progenitor accretion discs. For example, Marcy et al. [2] have investigated two G-type main-sequence stars HD 210 277 and HD 168 443, and have concluded the existence of companions orbiting around these stars and having masses comparable with the mass of Jupiter. Their orbits are with large eccentricities: e = 0.45 and e = 0.54, respectively. Such eccentric orbits may result from gravitational perturbations imposed by other orbiting planets or stars, by passing stars in a dense star cluster, or by the eccentric protoplanetary disc. While HD 168 443 exhibits a long-term velocity trend, consistent with an undetected yet (directly) close stellar companion, HD 210 277 appears to be a single star. Consequently, the later possibility is most likely the explanation for its large eccentricity. This picture also takes a support from the widespread explanation of the superhump events in the light-curves of SU UMa type stars, that explores the eccentric structure of the discs. It is natural that under such circumstances the interest to the theory of accretion discs with non-circular orbits of their particles has increased during the recent years, because the explanation of the properties of such objects is a necessary condition for understanding the superhump events as a whole. The complexity of this problem may be illustrated by the comparison between the cataclysmic variables and low-mass X-ray binaries. As pointed out by Haswell et al. [3], in cataclysmic variables superhumps are believed to result from the presence of 3:1 orbital resonance in the accretion disc. Then the accretion disc becomes non-axisymmetric and precesses. The variations of luminosity in cataclysmic variables are caused by a tidally-driven modulation of the viscous dissipation into the disc, depending on the beat between the orbital and disc precession period. By contrast, in low-mass X-ray binaries the tidal dissipation in the outer parts of the accretion disc is unimportant because the optical emission is dominated by reprocessing of the X-rays emitted from the compact object. Consequently, in these two cases the superhump modulation is caused by two distinct mechanisms. Similarly, detailed hydrodynamic simulations of the superoutburst events in dwarf novae, including the full tidal potential of the binary system, are performed in the work of Truss et al. [4]. Their theoretical (using numerical methods) investigations of the mass flux through the disc, the growth rate of the superhumps and the disc eccentricity show that the superoutburst-superhump phenomenon is a direct result of tidal instability. Other studies of such events demonstrate that the stabilization of the superhump period at low values favours model, in which period changes arise strictly from eccentricity changes rather than mean radius changes in the disc [5]. This explains why decreasing period and decreasing amplitude are strongly linked in the superhumps of dwarf-nova.

Other observational studies of the dwarf nova WZ Sagittae [6] whose eclipses permit measuring the location and brightness of the mass-transfer hot spot imply that the disc must be very eccentric and nearly as large as the white dwarf's Roche lobe. Because the hot-spot luminosity exceeds its quiescent value by a factor of up to 60, this indicates that the enhanced mass transfer from the secondary plays a major role in the eruption, determining the geometrical shape of the accretion disc as well.

Accretion discs are expected to occur in a large number of various selestial objects, for example around protostars, accreting compact objects on stellar binary systems and also around supermassive black holes at the cores of galaxies. The importance of these phenomena for astrophysics lies in the circumstance that they are related with local hydrodynamical or magnetohydrodynamical processes which can allow outward transport of the angular momentum of the infalling matter. So, a considerable part of the surrounding matter is able to reach the surface of the compact object, increasing its mass and changing its spin velocity. The significance of the above properties naturally explains why a large number of theoretical (both analytical and numerical) models of accreting discs have been developed during the last two decades and have been compared with the available observational astronomical data. A lot of theoretical works have revealed many of the subtle properties of the accretion discs in order to accommodate the models to these data. In particular, Ogilvie [7] stresses that an important and widely neglected aspect of the interaction between an accretion disc and a massive companion with a coplanar orbit is the vertical component of the tidal force. The response of the disc to the vertical forcing is resonant at certain radii, at which a localized torque is exerted and at these radii a compressive wave may be emitted. The m = 2 inner vertical resonance in a binary star is typically located within the tidal truncation radius of a circumstellar disc. This resonance contributes to angular momentum transport and produces a potentially observable non-axisymmetric structure. Larwood and Kalas [8] numerically investigate a close stellar fly-by encounter of a symmetrical circumstellar planetesimal disc and derive that this mechanism can give rise to the many kinds of asymmetries and substructures attributed to the edge-on dusty discs of β Pictoris. Their conclusions are supported by the optical coronographic observations of the outer perts of the disc of β Pictoris whose asymmetry was found to be approximately 25 %.

Eclipses in the binary stellar systems are often used techniques for obtaining numerical estimates of the perameters of these objects, including the characteristics of accretion discs when they are present in such binaries. For example, the use of the hot-spot eclipse times of the deeply eclipsing dwarf nova IY UMa enables to trace out the shape of its disc during the late superhump era. The result is an eccentric disc [9]. The analysis of the high-speed photometry of the dwarf nova EX Draconis through its outburst cycle reveals that the disc expands during the rise phase until it fills the most of the primary Roche lobe and one-armed spiral structure present in the disc at the stages of the outburst [10].

The above mentioned papers are only a small part of the numerous theoretical and observational evidence illustrating that the accretion disc may have not only eccentric shape, but also a complicated internal structure like gaps and spiral density waves. This situation makes reasonable the studying of discs composed by particles moving on *elliptical* orbits around a compact gravity center. In the present work we continue an earlier investigation [11] of an accretion disc model developed by Lyubarskij et al. [12]. Their analysis of the accretion flow appears to some extent as a generalization of the standard α-disc accretion [13] to the case of non-circular (i.e., elliptical) orbits. Here, our goal is to obtain analytical solutions to the dynamical equation describing the accretion flow according to the model of Lyubarskij et al. [12] and to derive the domain where our results are valid. For the latter reason it is worthy to note also some of the limitations of the α-disc model of Shakura and Sunyaev [13]. The standard model of disc accretion assumes that the gravitational energy is locally efficiently radiated from both sides of the outer disc surface and the gas keeps its (nearly) *Keplerian* rotation because the interactions between the neighbouring radial annuli are neglected. However, there may exist an important process which leads to a structure different from that picture – namely, the advection. Physically, the advection process means that the generated energy via viscous dissipation is restored as entropy of the accreting gas rather than being radiated. As stressed in [14], the advection effect may be very important both for the cases of low and high rates, since radiation decreases efficiently under these circumstances. The angular velocity of the gas is much lower than the Keplerian, i.e. the sub-Keplerian velocity is one of the general properties of the advection-dominated flows. In the model of an optically-thick disc considered in [14] , the emission of blackbody radiation from the disc surface is so inefficient that the advection cooling dominates over surface cooling because of the high accretion rate that leads to photon trapping in the disc. While in the standard model of Shakura and Sunyaev [13] in the radiation-pressure dominated region thermal instability exists, in the advection-dominated case the accretion flow is thermally stable in the same range. The reason why the advection cooling stabilizes the radiation-pressure dominated region of the disc is that it plays two important roles: balancing and lowering the generated energy.

Another important property of the standard α-disc model is that turbulent stresses leading to outward angular momentum transport in accretion discs are treated as resulting from isotropic effective viscosity, related to the pressure through the α-parametrization of Shakura and Sunyaev [13]. This simple approach may be adequate for the simplest aspects of accretion disc theory and was historically necessitated by an incomplete understanding of the origin of turbulence [15]. Recently Balbus and Hawley [16-19] have shown that the magnetorotational instability provides a mechanism of generating turbulent Reynolds and Maxwell stresses in sufficiently ionazed discs for which the α-viscosity model is not able to provide satisfactory description of many aspects of this process. The new generation of models taking into account these properties of the accretion flows should be particularly useful in understanding the dynamics of warped, eccentric and tidally distorted discs and also non-Keplerian flows (which are expected, for example, close to black holes).

In the above mentioned notes we have touched some of the unresolved problems of the standard α-disc accretion model. These deficiencies must be kept in mind when the results obtained in the next sections of the present paper are considered. Our solutions to the particular cases of the dynamical equation, governing the eccentric accretion flow, are in fact, solutions of a problem treating the accretion picture on the base of the Shakura-Sunyaev model for circular discs [13], extended to the case of eccentric orbits by Lyubarskij et al. [12]. All the restrictions concerning the applicability of such theories in reality (tested by means of observations) will be valid for our solutions even when they are mathematically exact in the considered domain of parameter space).

Numerical Solutions to the Dynamical Equation of the Accretion Flow

In a previous paper [11] we have obtained the explicit form of the dynamical equation valid for a *stationary* accretion, as considered by Lyubarskij et al. [12]. These results are obtained for particular values of the exponent n (namely, n = -1, 0, 1, 2, 3) in the accepted power-law relation $\eta = \beta \Sigma^{n}$ between the viscosity η and the surface density Σ of the eccentric accretion disc. Following the notations in [11], this equation can be written as a homogeneous second order ordinary differential equation:

(1) $\quad \mathrm{A}(e, \dot{e}, n)\, \ddot{e} + \mathrm{B}(e, \dot{e}, n)\, \dot{e} = 0$,

where $A(e, \dot{e}, n)$ and $B(e, \dot{e}, n)$ are already known functions of e, $\dot{e}$ and n (their derivation is the main outcome of [11]) and the dot (.) denotes differentiation with respect to $u \equiv \ln p$; p is the focal parameter of elliptical trajectories of the gas particles. In their paper Lyubarskij et al. [12] have numerically solved equation (1) for some particular values of n (see Figs. 2-5 from [12]) and have obtained three classes of solutions. In order to verify our analytical derivations we have also repeated the solution of differential equation (1) using a numerical method and have found agreement between our graphics and the graphics in [12]. We shall further use the exact results of the numerical integration of dynamical equation (1) as standards with respect to which we shall compare the validity of our analytical approximations to the solutions of (1). This will give us the opportunity to establish the domain where our analytical approach is successful and also to estimate the precision of the approximations. We shall test the most simple and the most suitable for the analytical applications fitting of the eccentricity dependence $e = e(u \equiv \ln p, n)$ – the polynomial approximation.

Polynomial Approximation to the Solutions of the Dynamical Equation

We shall try to find a solution to equation (1) using the following power-law expansion for the unknown eccentricity function $e = e(u, n)$

$$e(u, n) = \sum_{i=0}^{M} a_i(n)\, u^i\,, \tag{2}$$

where the coefficients $a_i(n)$ (i = 0, ..., M) are unknown functions on the parameter n and are subject to further determination. Because our investigation of the problem of finding solutions to equation (10) is restricted only to five fixed integer values of n (n = -1, 0, 1, 2, 3), in what follows we shall omit the explicit notation of the dependence on n. The meaning of n will be clear from the heading of the considered case. Generally speaking, the power series (2) may contain an infinite number of terms (i. e., $M = \infty$). But in order to obtain practically effective and usable computational procedure, we must truncate series (2), assuming some finite value M (i. e., $M < \infty$). In the present work we have confined ourselves to the value M = 5. This was done for computational reasons. When we attempted to determine higher order coefficients a_6, a_7, etc., the analytical expressions for these functions became so long and complicated that the available memory of the computer was not enough in order to perform the analytical evaluation of these quantities. Such a circumstance must not be considered only as a technical problem, but also as an evidence that the analytical approach we have selected is not productive in view of high accuracy evaluation of the eccentricity $e(u)$ along the radius of the accretion disc. Consequently, the applied method for approximation of $e(u)$ by means of a polynomial is working effectively only when the truncation M = 5 is appropriate to give the desired precision. We stress that the main aim of our investigation is to find analytical expressions for $e(u)$ which are suitable for further analytical manipulations and which should not produce too complicated mathematical formulae as final results. Unfortunately, if the lower order coefficients a_2, a_3, a_4 and a_5 are tedious expressions, there is not any reasonable hope to expect that the application of the approximation (2) will ensure this optimistic outcome. So, we have to confine ourselves to the more particular case of simplifying of the task, namely, to use in the intermediate analytical calculations the suitable for analytical work polynomial representation (2), without introducing in it the explicit form of the coefficients a_i (i = 0, 1,..., M) and only arriving at the final results to do these replacements.

In the case $n = 1$, when dynamical equation (1) has relatively simple form (see eq. (14) from [11]) we have made comparison for two different cut-offs of series (2): for M= 5 and M = 8 (a better degree of approximation). The results show that we cannot establish an evident increase of accuracy when M = 8 is used instead of M = 5. Transferring this conclusion to the cases $n = 0$, 1, 2 and 3 (without an *explicit* proof !), we are challenged to believe that assuming the cut-off M = 5 is a reasonable compromise between the complexity of the coefficients a_i (i = 0, 1,..., M) and the accessibility of a higher accuracy of the approximation (2) by means of greater values of the cut-off M. Taking in advance the results in the next section of this chapter, we give the "size" of the coefficients a_i (i = 2,..., M_5(or M_8)) as evaluated by the occupied computer memory. Of course, this is a very rough measure of the complexity of these quantities, but their explicit formulae are too long and complicated to be given in this paper. For this reason, we prefer to prepare only a brief sketch of their length, which of course does not reveal their internal structure. Such a description of the coefficients a_i (i = 2,..., M_5 (or M_8).) may seem as a very fictive picture of their real mathematical properties. Nevertheless, the data clearly demonstrate the increasing complication of the computational procedure when higher order terms are taken into account into power series (2).

It also supports the unavoidable need to cut off this series at M = 5. Consequently, the derived domain of validity of approximation (2) can be hardly extended (for a value of the accuracy fixed in advance) of the solution $e = e(u, n)$) to a wider region in the parameter space of the initial conditions ($e_0 = e(u_0 \equiv \ln p_0, n)$, $\dot{e}_0 = \dot{e}_0(u_0 \equiv \ln p_0, n)$). We stress that we have chosen to investigate/derive the solutions to dynamical equation (1) by using polynomial approximation (2) for its simplicity in view of its application to analytical evaluations (easy differentiation and integration, simple procedures for finding roots and singularities, etc.). However, we *do not state* that there are not more complicated *exact* analytical solutions to the homogeneous second order ordinary differential equation (1). Our inability to find such solutions $e = e(u \equiv \ln p, n)$, though they may be very awkward, does not mean that they cannot be found at all ! Of course, we shall confine ourselves not to the maximal purpose to obtain these solutions, but to put into use more complicated approximate expressions than (2), in order to expand the domain of validity of the approach. However, this eventually may not be the successful position. Out of the complicated analytical applications, the more exact solutions (in the sense of an extended domain of validity) do not ensure in advance a better coincidence with the observations. This is because of the inaccuracy of the physical description of the reality inherent to the model of Lyubarskij et al. [12], as discussed earlier. Therefore, the use of more precise approximations than (2) does not guarantee that the higher price to be paid for that is an acceptable decision. We only mention in that sense, that we have tried to transform the most simple of the considered dynamical equations (namely, eq. (14) from [11]; Case $n = -1$) substituting $e(u, n = -1) = \cos[\psi(u, n = -1)]$, but the introduction of the angular coordinate $\psi(u, n)$ did not produce the expected result – the equation was not reduced to a form supposing an easier to find exact solution.

An important note should be made. The general solution of differential equation (1) depends on two integration constants which are subject to determination from the initial and boundary conditions. The considered problem deals with a *stationary* accretion disc, so it is sufficient to give the values of two physical characteristics of the disc at a given fixed value of the focal parameter p_0 (respectively, $u_0 \equiv \ln p_0$). The most natural choice seems to select the eccentricity e and its derivative $\dot{e} \equiv \partial e / \partial u \equiv \partial (\ln e) / \partial p$. In terms of the polynomial approximation used here, the later variable may be written as

$$(3) \qquad \dot{e}(u,n) = \sum_{i=0}^{M-1} (i+1)\, a_{i+1}\, u^i \;; \quad (n = -1, 0, 1, 2, 3).$$

In what follows in this paper, we choose the value of the focal parameter at which we give the boundary conditions, to be equal to 1. Respectively, $u_0 \equiv \ln p_0 = 0$. Because $p = b^2/a$, this means that we set the boundary conditions on an ellipse with major and minor semiaxes a and b, respectively, satisfying the relation $b = a^{1/2}$. Then, omitting the notation of n, from (2) and (3) we obtain the simple relations $e_0 \equiv e(u_0) = a_0$ and $\dot{e}_0 \equiv \dot{e}(u_0) = a_1$. In other words, using the above gauge, we in fact assign values to the coefficients a_0 and a_1. Consequently, these quantities must be considered as independent *input* parameters of the problem and all other variables are functions of them. In particular, the higher order coefficients a_i (i = 2,..., M_5 (or M_8)) also depend on $a_0 = e_0$ and $a_1 = \dot{e}_0$ (and on the exponent n in the viscosity law $\eta = \beta\, \Sigma^n$, of course). For each fixed value n = -1, 0, ..., 3, we have derived a_i (i = 2,..., M_5 (or M_8)) in an explicit form as functions on a_0 and a_1. And then the lengths of these expressions were approximately evaluated as shown by the data on Table 1.

Table1. Approximate evaluations of the "size" of the coefficients $a_i\,(e_0, \dot{e}_0, n)$.

Powers / Coefficients	n = -1	n = 0	n = +1	n = +2	n = +3
a_2	3.22 kb	8.68 kb	14.3 kb	6.13 kb	8.16 kb
a_3	3.83 kb	46.4 kb	111 kb	28.7 kb	40.2 kb
a_4	6.38 kb	143 kb	369 kb	76.7 kb	127 kb
a_5	9.04 kb	2.31 Mb	856 kb	168 kb	281 kb
a_6	12.07 kb				
a_7	19.0 kb				
a_8	28.7 kb				

Selecting the polynomial approximation (2) and the boundary conditions $e_0 \equiv e(u_0 = 0)$ and $\dot{e}_0 \equiv \dot{e}(u_0 = 0)$, we can also compute easily the second derivative of the eccentricity:

$$(4) \qquad \ddot{e}(u) = \sum_{i=0}^{M-2} (i+2)(i+1)\, a_{i+2}\, u^i \ ,$$

the difference $e(u) - \dot{e}(u)$ between the eccentricity $e(u)$ and its derivative $\dot{e}(u)$:

$$(5) \qquad e(u) - \dot{e}(u) = \sum_{i=0}^{M-1} [a_i - (i+1)\, a_{i+1}]\, u^i \ ,$$

the powers of the eccentricity $e(u)$ and $\dot{e}(u)$:

$$(6) \qquad e^2(u) = \sum_{i=0}^{M} \Big(\sum_{k=0}^{i} a_k\, a_{i-k} \Big)\, u^i \ ,$$

$$(7) \qquad e^3(u) = \sum_{i=0}^{M} \Big(\sum_{k=0}^{i} a_k \sum_{p=0}^{i-k} a_p\, a_{i-k-p} \Big)\, u^i \ ,$$

$$(8) \qquad e^4(u) = \sum_{i=0}^{M} \Big[\sum_{l=0}^{i} \Big(\sum_{k=0}^{l} a_k\, a_{l-k} \Big) \Big(\sum_{k=0}^{i-l} a_k\, a_{i-l-k} \Big) \Big]\, u^i \ \text{, etc. ,}$$

$$(9) \qquad \dot{e}^2(u) = \sum_{i=0}^{M} \Big[\sum (k+1)(i-k+1)\, a_{k+1}\, a_{i-k+1} \Big]\, u^i \ ,$$

$$(10)\ \dot{e}^3(u) = \sum_{i=0}^{M} \Big\{ \sum_{k=0}^{i} \Big[\sum_{l=0}^{k} (l+1)(k-l+1)\, a_{l+1}\, a_{k-l+1} \Big] (i-k+1)\, a_{i-k+1} \Big\}\, u^i, \ \text{etc.}$$

Obviously, the higher powers of the eccentricity $e(u)$ and its derivative $\dot{e}(u)$ become more and more bulky, but nevertheless they may be computed in an explicit manner. We shall not, of course, give them here. But we shall write a formula, also used in our computations. According to [20] (see expression 5.3):

$$(11)\quad (1-e^2)^{1/2} = \sum_{i=0} d_i\, e^{2i}\ ;\ d_0 = 1\ ,\ d_1 = -\tfrac{1}{2}\ ;$$

$$d_i = -\,[(2i-3)\,!!]\,/\,(2i)\,!! \quad \text{for } i = 2, 3, \ldots\ ;$$

(this means that $d_2 = -1/8$, $d_3 = -1/16$, $d_4 = -5/128$, etc.).

The above formula is also applied for expansions in powers of the expressions $(1-\dot{e}^2)^{1/2}$ and $[1-(e-\dot{e})^2\,]^{1/2}$, replacing $e(u)$ by $\dot{e}(u)$ or by $[e(u)-\dot{e}(u)]$, respectively. Combining the expansions like written above, we have also computed the power series expansions of $(1-e^2)^{3/2}$, $(1-e^2)^{5/2}$, …; $(1-\dot{e}^2)^{3/2}$, $(1-\dot{e}^2)^{5/2}$, … ; $[1-(e-\dot{e})^2\,]^{3/2}$, $[1-(e-\dot{e})^2\,]^{5/2}$, … . As a final result, we have obtained the necessary power series expansions of the coefficients $A(e, \dot{e}, n)$ and $B(e, \dot{e}, n)$ for every fixed value of $n = -1, 0, 1, 2$ and 3. Substituting these into dynamical equation (1) and reducing after some algebraic manipulations its left-hand side to a series in powers of u, we transform (1) to the following form:

$$(12)\quad \sum_{i=0}^{M} c_i(e_0, \dot{e}_0, a_2, a_3, \ldots, a_M, n)\, u^i = 0\ ,\quad (n = -1, 0, 1, 2, 3\ ;\ M = 5 \text{ or } 8)\ .$$

This nullification must be fulfilled for arbitrary values of u and, consequently, all the coefficients $c_i(e_0, \dot{e}_0, a_2, a_3, \ldots, a_M, n)$ also must be equal to zero:

$$(13)\quad c_i(e_0, \dot{e}_0, a_2, a_3, \ldots, a_M, n) = 0\ ,\ (n = -1, 0, 1, 2, 3\ ;\ M = 5 \text{ or } 8).$$

It should be mentioned that multiplying different kinds of expressions like these given by formulae (2) – (11), we shall obtain terms including powers of u greater than M. Such circumstance leads to undesirable complications of the intermediate calculations of the left-hand side of (12). For example, we may get terms proportional to u^{15}, u^{16}, u^{17}, etc. To avoid this objectionable situation, we cut off the powers of u which are much greater than M (M = 5 or 8). Of course, these manipulations of the intermediate analytical expressions must be done carefully in order to avoid in the final sum erroneous truncations of terms proportional to u^i ($i \le$ M). In that sense, we insured ourselves by formally preserving the terms in (12) up to order u^{M+3}, which do not create too much problems with respect to the complexity of the formulae.

Deriving explicitly the coefficients $c_i(e_0, \dot{e}_0, a_2, a_3, \ldots, a_M, n)$, it turns out that for each fixed value of $n = -1, 0, 1, 2$ and 3, relations (13) give linear equations for a_i (i = 2, 3,..., M = 5 or 8) provided that the low order coefficients a_i are already known. That is to say, equalities (13) are in fact recurrence formulae for a_i (i = 2, 3,..., M = 5 or 8). In more details, we have that the free term of (11) depends only on e_0, $\dot{e}_0$, a_2 and n, but not on a_3, a_4, etc. Equating it to zero

(14) $\quad c_0(e_0, \dot{e}_0, a_2, n) = 0$,

we obtain *a linear* equation for the unknown quantity a_2 , which is easily solved. Further we see that the coefficient c_1 depends only on e_0, $\dot{e}_0$, a_2, a_3 and n. Similarly, it turns out that its nullification

(15) $\quad c_1(e_0, \dot{e}_0, a_2, a_3, n) = 0$

appears as *a linear* equation for the unknown quantity a_3 under the condition that the value of a_2 is already computed from (14). This sequence of operations can be extended, because

$c_2 \equiv c_2(e_0, \dot{e}_0, a_2, a_3, a_4, n) = 0$, $c_3 \equiv c_3(e_0, \dot{e}_0, a_2, a_3, a_4, a_5, n) = 0$, etc., and these relations are *linear* equations for a_4, a_5, etc. realizing all these steps, we obtain finally the polynomial approximation (2) for the eccentricity $e(u)$ of the orbits of the accretion disc particles. We again stress that (2) is only *a fitting* to the exact solution of the dynamical equation (1), regardless of whether the latter is obtained by numerical or (may be ?) analytical methods.

It remains to check the precision of our approach, comparing the graphics of the solutions of type (2) with the results of the numerical integration of (2). It is clearly seen that in the parameter space $(e, \dot{e})$ (for every fixed value of n) there are regions where the polynomial approximation (2) gives an excellent agreement with the exact solutions of equation (1). There the difference $e(u)_{\text{exact}} - e(u)_{\text{polynomial}}$ may be less than 10^{-6} or even better!). But there are also domains where it is fully unacceptable. The transition between these two regions is, however, too steep as a rule. Consequently, it is very important to determine precisely the boundaries of the validity domain of the tested (in this paper) truncated power-low series approximation (2). We remind that the eccentricity $e(u)$ and its derivative $\dot{e}(u)$ must satisfy the following restrictions for all u: $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ [11], which in turn determine the shape of the overall domain, where we are seeking solutions of dynamical equation (1). The above equalities will be discussed in more details in a forthcoming paper. We also note that (for our illustrative purposes) we have chosen to reject all the polynomial solutions at the level where

$|e(u)_{exact} - e(u)_{polynomial}| \le 0.1$, i.e., we accept that satisfactory accuracy is achieved when the deviation of the tested analytical approximation (2) from the exact (numerically obtained) solution is better than 0.1. It is evident that the domain where polynomial approximation (2) turns out well has a complex shape, which we shall not try to evaluate analytically. But nevertheless, this domain is large enough to say that approach (2) makes sense and the considered problem is solved at least particularly.

Singularities of the Power - Law Expansion Coefficients a_i

Evaluating the applicability of truncated (M = 5 or M = 8) power series (2) relative to the possibility to solve the equation governing the stationary accretion flow in the model of Lyubarskij et al.[12], we have to study in more details the behaviour of coefficients a_i (i = 2, 3,..., M) in the parameter space $(e, \dot{e})$ for each n = -1, 0, 1, 2 and 3 in separate. Except for the case n = -1, the explicit expressions for these quantities (functions on e and $\dot{e}$) are so long and complicated, as discussed earlier, that we are forced to apply graphical argumentation rather than purely analytical one. Our aim is not to investigate these coefficients as a whole, but only their denominators. The nullifications of the latters dangerous sources, generating divergences of series (2). Fortunately, this problem is essentially simplified by the property that the denominators of a_3, a_4, a_5,..., a_8 are integer powers of the denominator of a_2 within to a non-zero factor. This will be explicitly shown in the following example, illustrating the case n = -1. Therefore, it is enough to concentrate our efforts on finding the roots of the denominator of a_2.

The case n = -1 is the most simple situation. For n = -1 the dynamical equation is relatively simple in comparison with the other cases ([11], see eq. (14)]:

(16) denominator $a_2 = \mathrm{A}(n = -1)$,

where $\mathrm{A}(n = -1) = (1 - e^2)(144 - 80e^2 - 16e^4 - 8e^6 - 5e^8)$. This expression does not depend on $\dot{e}(u)$. Provided that $|e(u)| < 1$ for all u, the roots of the denominator a_2 (including their multiplicity) are:

$$
(17) \left[\begin{array}{l} e_{1,2} = - e_{3,4} = - 1.209\,912 \ \pm\ 1.138\,902\, i \ ; \\ e_{5,6} = \pm\, 1.097\,309 \ ; \quad e_{7,8} = \pm\, 1 \ ; \\ e_{9,10} = - 4.422 \times 10^{-17} \pm 1.771\,346\, i \ , \end{array} \right.
$$

where i is the imaginary unit. Obviously, there are not troubles generated by the nullification of the denominator of a_2, because all the roots are complex or greater than (or equal to) 1 by absolute value. The higher order coefficients have denominators as follows:

(18) denominator $a_3 = 96 \times [A(n = -1)]^2$,

(19) denominator $a_4 = 1536 \times [A(n = -1)]^3$,

(20) denominator $a_5 = 30720 \times [A(n = -1]^4$, etc.

In *all* the coefficients a_i (i = 2, 3,…, 8), the denominators are multiple to the powers of $A(n = -1)$ and the other multipliers in each factorization are not equal to zero. In the other cases n = 0, 1, 2 and 3, the situation is not so *analytically* clear because of the complexity of the coefficients a_i (i = 2, 3,…, 8), but the *graphical* representations of their denominators indicate that there is no evidence of nullification of the latters (*at least*) for extended domains in the plane $(e, \dot{e})$. Of course, the domains of validity of polynomial approximation (2) must agree with the restrictions $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ [11], as mentioned earlier.

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ТЪНКИ ВИСКОЗНИ ЕЛИПТИЧНИ АКРЕЦИОННИ ДИСКОВЕ С II. ПОЛИНОМНИ РЕШЕНИЯ НА ДИНАМИЧНОТО УРАВНЕНИЕ ЗА ЦЕЛОЧИСЛЕНИ СТОЙНОСТИ НА СТЕПЕНИТЕ В ЗАКОНА ЗА ВИСКОЗИТЕТА

Д. Димитров

Резюме

За целочислени степенни показатели n (n = -1, 0, 1, 2 и 3) в закона за вискозитета $\eta = \beta\ \Sigma^n$, където Σ е повърхностната плътност на акреционния диск, ние сме изследвали полиномния подход за намиране

решенията на динамичното уравнение в модела на акреционен диск на Любарски и др.[12]. Разложенията в степенни редове на ексцентрицитета $e(u)$ и неговите производни $\dot{e}(u)$ и $\ddot{e}(u)$, техните степени $e^2(u)$, $e^3(u)$,...; $\dot{e}^2(u)$, $\dot{e}^3(u)$ и т.н. са орязани при подходящи стойности на степенните показатели и след това те са поставени в динамичното уравнение. Извършвайки допълнителни орязвания в промеждутъчните произведения и резултати, въпреки това ние сме достигнали точност на решението по-добра от ~ 10% за една достатъчно голяма област в равнината $(e, \dot{e})$.

Тези резултати са установени графически чрез сравняване на полиномната апроксимация на ексцентрицитета $e_{\text{polynomial}}(u)$ (където $u \equiv ln\, p$, p е фокалният параметър на елипсата) и точните стойности $e_{\text{exact}}(u)$ получени посредством числено решаване на уравнението. Решенията на уравнението на движение, което е обикновено диференциално уравнение от втори ред, са параметризирани с помощта на граничните условия $e_0 \equiv e(u = 0)$ и $\dot{e}_0 \equiv \dot{e}(u = 0)$. Коефициентите в разложенията в степенни редове са оценени в явен вид, но поради тяхната сложност ние даваме само техните "размери", разглеждани като дължини на файловете които ги представят. Дискутирани са също проблемите отнасящи се до възможните сингулярности на резултатите.